§5.3 The relationship between measure theoretic entropy and topological entropy.

- Recall that for a TDS (X, T),

$$h(T) = \sup_{\alpha} h(T, \alpha)$$

where α ranges over all open covers of X,

$$h(T, \alpha) = \lim_{n\to\infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \right).$$

where $N(\alpha)$ = the smallest number of members of α that can cover X.

Notice that for two open covers α, β of X,

we write $\alpha < \beta$ if each member of β is the subset of a member of α.

Clearly $N(\alpha) \leq N(\beta)$ if $\alpha < \beta$.

- Lem 5.4. Let (X,T) be a TDS, and let (α_n) be a sequence of open covers of X with $\operatorname{diam}(\alpha_n) \to 0$.

  Then $$h(T) = \lim_{n\to\infty} h(T,\alpha_n).$$

Pf. Let α be an open cover and let $\delta>0$ be a Lebesgue number (i.e. any set $A\subset X$ with $\operatorname{diam} A<\delta$ is the subset of a member in α).

Now if α_n is an open cover of X with $\operatorname{diam}(\alpha_n)<\delta$,

then $\alpha < \alpha_n$.

Hence $$\bigvee_{i=0}^{k-1} T^{-i}\alpha < \bigvee_{i=0}^{k-1} T^{-i}\alpha_n \quad \text{for all } k\in\mathbb{N},$$

which implies that

$$N\Big(\bigvee_{i=0}^{k-1} T^{-i}\alpha\Big) \leq N\Big(\bigvee_{i=0}^{k-1} T^{-i}\alpha_n\Big)$$

and so

$$h(T,d) \leq h(T,d_n).$$

Hence $\quad h(T,d) \leq \underline{\lim}_{n\to\infty} h(T,d_n)$

Thus

$$h(T) \leq \underline{\lim}_{n\to\infty} h(T,d_n) \leq \overline{\lim}_{n\to\infty} h(T,d_n) \leq h(T).$$

$\boxtimes$

- The following fundamental thm was due to Goodwyn in 1968

Thm 5.5. Let (X,T) be a TDS. Then

$$h(T) = \sup\{ h_\mu(T) : \mu \in M(X,T) \},$$

where $M(X,T)$ denotes the collection of T-invariant Borel. prob. measures on X.

Recall that $\quad h_\mu(T) = \sup_{\xi} h_\mu(T,\xi)$

where d ranges over all partitions of X,

$$h_\mu(T,\xi) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{k-1} T^{-i}\xi \right).$$

To prove Thm 5.5, we need the following Lemmas.

Lem 5.6. Let X be a compact metric space and $\mu \in \mathcal{P}(X)$.

(1) If $x \in X$ and $\delta > 0$, then $\exists\ 0 < \delta' < \delta$ such that
$$\mu(\partial B(x, \delta')) = 0.$$

(2) If $\delta > 0$, $\exists$ a partition $\{A_1, \cdots, A_k\}$ of $(X, \mathcal{B}(X))$ such that $\operatorname{diam}(A_i) < \delta$ and $\mu(\partial(A_i)) = 0$.

Pf. (1) It is clear since we don't have an uncountable collection of disjoint sets of positive measures.

(2) By (1), we can find a finite open cover
$$\{U_1, \cdots, U_m\} \text{ of } X \text{ by balls of radius } < \frac{\delta}{2}$$
and $\mu(\partial U_i) = 0$.

Set $A_1 = \overline{U_1}$, $A_2 = \overline{U_2} \setminus \overline{U_1}$, $A_3 = \overline{U_3} \setminus (\overline{U_1} \cup \overline{U_2})$, $\cdots$

Then $\partial(A_i) \subset \bigcup_{j=1}^{m} \partial U_j$. Hence $\mu(\partial A_i) = 0$.

(Check: Let $x \in \partial(A_i)$, suppose that $x \notin \bigcup_{k=1}^{m} \partial(U_k)$.

Then $x \in U_i$ and $d(x, U_j) > 0$ for all $j < i$. (see Fig 1).

Hence $x \in \operatorname{int}(A_i)$.)

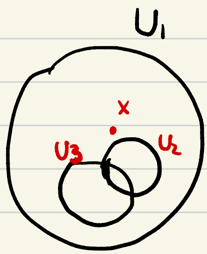

Fig 1

Clearly $\{A_1, \cdots, A_m\}$ is a Borel partition of X.

$\blacksquare$

Proof of the upper bound : $h_\mu(T) \leq h(T)$ for all $\mu \in M(X,T)$.

Pf. Let $\mu \in M(X,T)$. Let $\xi = \{A_1, \cdots, A_k\}$ be a finite Borel partition of X. Choose $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{k \log k}.$$

For each i,

Choose compact $B_i \subset A_i$ with $\mu(A_i \setminus B_i) < \varepsilon$.

Take $B_0 = X \setminus \bigcup_{i=1}^{k} B$. Then B_0 is open.

Moreover $\beta = \{B_0 \cup B_1, \ B_0 \cup B_1, \cdots, B_0 \cup B_k\}$ is an open cover of X.

Let $\eta = \{B_0, B_1, \cdots, B_k\}$. Then η is another partition of X.

Moreover

$(\phi(x) := -x\log x)$.

$$H_\mu(\xi|\eta) = \sum_{i=0}^{k}\sum_{j=1}^{k} \mu(B_i)\,\phi\left(\frac{\mu(B_i\cap A_j)}{\mu(B_i)}\right)$$

$$= \mu(B_0)\cdot\sum_{j=1}^{k}\phi\left(\frac{\mu(B_0\cap A_j)}{\mu(B_0)}\right)$$

(by Jensen's inequality)

$$\leq \mu(B_0)\cdot\log k$$

$$\leq \varepsilon\, k\log k$$

(since for $i\neq 0$, $\frac{\mu(B_i\cap A_j)}{\mu(B_i)} = 0$ or 1 for all j)

Hence $h_\mu(T,\xi) \leq h_\mu(T,\eta) + H(\xi|\eta)$

$\leq h_\mu(T,\eta) + 1$. (1)

Next we compare $h_\mu(T,\eta)$ and $h(T,\beta)$.

Notice that each member in $\bigvee_{i=0}^{n-1} T^{-i}\beta$

is of the form $(B_0\cup B_{i_1})\cap T^{-1}(B_0\cup B_{i_2})\cap\cdots\cap T^{-(n-1)}(B_0\cup B_{i_n})$

which intersects at most 2^n many members

in $\bigvee_{i=0}^{n-1} T^{-i}\eta$.

Hence $$N\left(\bigvee_{i=0}^{n-1} T^{-i}\beta\right) \geq \frac{N\left(\bigvee_{i=0}^{n-1} T^{-i}\eta\right)}{2^n}$$

But
$$\log N\left(\bigvee_{i=0}^{n-1} T^{-i}\eta\right) \geq H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\eta\right)$$

(add a simple lem about this).

Hence
$$H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\eta\right) \leq \log\left(N\left(\bigvee_{i=0}^{n-1} T^{-i}\beta\right)\cdot 2^n\right)$$

It follows that
$$h_\mu(T,\eta) \leq h(T,\beta) + \log 2.$$

This combined with ① yields

$$\begin{aligned} h_\mu(T,\xi) &\leq h_\mu(T,\eta) + H_\mu(\xi|\eta) \\ &\leq 1 + h(T,\beta) + \log 2 \\ &\leq h(T) + 1 + \log 2. \end{aligned}$$

In the above inequality, replacing T by T^n, and ξ by $\bigvee_{i=0}^{n-1} T^{-i}\xi$ gives

$$h_\mu\left(T^n, \bigvee_{i=0}^{n-1} T^{-i}\xi\right) \leq h(T^n) + 1 + \log 2.$$

It is easy to check that

$$h_\mu\left(T^n, \bigvee_{i=0}^{n-1} T^{-i}\xi\right) = n\, h_\mu(T, \xi)$$

and also

$$h(T^n) = n\, h(T)$$ (see the justification after the proof the upper bound.)

Hence

$$n\, h_\mu(T, \xi) \leq n\, h(T) + 1 + \log 2,$$

i.e. $$h_\mu(T, \xi) \leq h(T) + \frac{1+\log 2}{n}.$$

Letting $n \to \infty$ gives $h_\mu(T, \xi) \leq h(T)$.

Hence $$h_\mu(T) = \sup_{\xi} h_\mu(T, \xi) \leq h(T).$$

This proves the upper bound. $\blacksquare$

Lem 5.7. Let (X, T) be a TDS. Then for each $n \in \mathbb{N}$,

$$h(T^n) = n\, h(T).$$

Pf. Let α be an open cover of X and $n \in \mathbb{N}$.

Then $\quad \alpha < \bigvee_{i=0}^{n-1} T^{-i}\alpha.$

Hence

$$h(T^n, \alpha) \leq h\left(T^n, \bigvee_{i=0}^{n-1} T^{-i}\alpha\right)$$

$$= n\, h(T, \alpha)$$

$$\leq n\, h(T).$$

Hence

$$h(T^n) = \sup_{\alpha} h(T^n, \alpha) \leq n\, h(T).$$

On the other hand,

$$n\, h(T, \alpha) = h\left(T^n, \bigvee_{i=0}^{n-1} T^{-i}\alpha\right) \leq h(T^n)$$

Hence

$$n\, h(T) = \sup_{\alpha} n\, h(T, \alpha) \leq h(T^n).$$

This proves that $\quad n\, h(T) = h(T^n)$. $\blacksquare$